Solving Equations: Students' Algebraic Thinking

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This study is part of current initiatives to extend the New Zealand Numeracy Development Projects from number into algebra. The paper describes an approach to linking numeracy with students' strategies for solving linear equations. Preliminary data from diagnostic interviews with approximately 450 year 7–10 students suggests that there is a hierarchy of sophistication of strategies. The most sophisticated strategy that a student is able to use is associated with the student's numeracy stage on the Number Framework.

Introduction

A new curriculum for all subject areas in New Zealand schools was launched on 6 November 2007 (Ministry of Education, 2007). The mathematics and statistics learning area is now divided into three strands rather than the previous six, with number and algebra being one strand. The achievement objectives of this strand are grouped to reflect the structure of the Number Framework (Ministry of Education, 2003), which details the number strategies that students use and the number knowledge required for these strategies. At the lower levels of the new mathematics and statistics learning area, the number and algebra achievement objectives are divided into: number strategies, number knowledge, equations and expressions, and patterns and relationships. The integration of number and algebra into one strand follows debate within the mathematics education community in New Zealand and within international research (see, for example, Carraher & Schiemann, 2007; Kieran, 1992; and Lee, 2001, for an explanation of algebraic thinking).

The Numeracy Development Projects (NDP) have been very successful at raising the achievement of New Zealand students in the number strand (Thomas & Tagg, 2007; Young-Loveridge, 2007), and various initiatives are currently underway to extend the projects into early algebra. Irwin and Britt (2007) examine the impact of the NDP on students' ability to generalise and suggest that students who can access a range of strategies to solve numerical problems are using thinking that is essentially algebraic in nature. These students are the ones who are likely to be able to work with algebraic symbols to express generality. University of Otago College of Education researchers involved in the study reported in this paper are examining students' strategies for solving linear equations and the relationship of these strategies to numeracy.

Background

Many students struggle with introductory algebra, and teachers have little to guide them in assisting their students to learn this important component of mathematics. Little is known about the effect of students' numeracy on the learning of early algebra or about the strategies that students use to solve equations. There is widespread agreement that few students easily understand algebra. The Cockcroft (1982) report highlights the fact that algebra is a source of considerable confusion and negative attitudes among students, while the title of Brekke's (2001) paper, "School algebra: Primarily manipulations of empty symbols on a piece of paper?" sums up the situation for many students.

Arithmetic in schools is often presented as a computation ready to complete, for example, 3 + 5 =. Students know that pressing the equals button on a calculator performs a calculation on whatever has been entered, so many students understand "equals" as meaning "compute now" rather than "is equivalent to" (Booker, 1987; Booth, 1988). Linchevski (1995) suggests that, in the transition from arithmetic to algebra, students need to move from a unidirectional view of the equals sign to a multidirectional one.

Closely related to this is the use of an equation as a process rather than as an object that can be operated on (Sfard, 1991). Students initially see equations as the description of an arithmetic process, for example, $2 \times 3 + 4 = x$, and when presented with an equation to solve, for example, $2x + 4 = 10$, they see this, too, as the description of an arithmetic process, with "guess and check" as the natural way of finding x. Even the more sophisticated strategy of solving the equation by working backwards may result from a view of equations as processes, yet this viewpoint is often not revealed until the students encounter equations of the kind $2x + 4 = 3x – 6$.

It is not possible to view this type of equation (unknowns on both sides) as the description of a process giving a result. What is essential is to view the equation as an object to be acted upon in order to be solved (Sfard & Linchevski, 1994). Herscovics and Linchevski (1994), however, present data that shows that many students revert to the strategy of "guess and check" to solve equations of this type. The issue of operating on unknowns is another important factor in understanding why equations with unknowns on both sides cause so many difficulties. Booker (1987) suggests that it is the shift from manipulation of numbers in order to solve an unknown to the manipulation of unknowns themselves that marks the entry into algebra proper.

A student's numeracy stage (Ministry of Education, 2003) is likely to be important for their understanding of expressions and equations (Irwin, 2003). Equations of the form $x + 3 = 7$ can be solved by advanced counters through guess and check, but they can be solved much more easily by part–whole thinkers, who are able to visualise 7 as 3 + 4. It can be argued that solving equations such as $3x = 15$ requires multiplicative thinking if a student is to do more than simply follow prescribed algorithms. Furthermore, equations of the kind $2x + 3 = 11$ might require an understanding of numbers beyond simple additive part–whole or multiplicative part–whole thinking.

The Current Study

As part of the NDP, diagnostic tools (including the Numeracy Project Assessment [NumPA] and Global Strategy Stage [GloSS]) have been developed to help assess students' numeracy stages. Also, during 2006, a group of researchers and teachers working on a Teaching and Learning Research Initiative (TLRI) project developed a diagnostic tool for assessing students' ability to solve linear equations (Linsell, Savell, Johnston, et al., 2006). The researchers in this study are now using all these tools as they investigate the links between numeracy and students' strategies for solving linear equations. During 2007, they interviewed approximately 450 year 7–10 students, with another 400 students interviewed early in 2008 before formal data analysis commenced. This paper explains the approach used in the study and some preliminary data.

In the 2006 TLRI study, the initial classification of strategies for solving equations was based on the work of Kieran (1992), whose review of the learning and teaching of algebra describes the strategies that students use. However, in this study, further strategies were added that students were observed using. For example, the use of an inverse operation (1c) (see Figure 1) on a one-step equation was considered to be different to Kieran's strategy of working backwards (3b) on multi-step equations. It was difficult to distinguish the difference between known basic facts (1a) and inverse operations (1c) because the students often justified their answers by describing the inverse operation, when, in fact, what they had done was use a known fact. One-step equations with larger numbers were therefore used to elicit the use of inverse operations.

The strategy of working backwards was found to be not as homogeneous as had been assumed. Many students partly worked backwards and then used either known facts (3c) or guess and check (3d). When large or decimal numbers precluded the use of these strategies, the students could no longer use the working backwards strategy (3b).

The strategy of using a diagram (5) was also included. This resulted from the study's explorations of questions in context, where a number of students solved equations through direct use of diagrams.

The final classification of strategies is listed in Figure 1. It should be noted that this list of strategies is not intended to be hierarchical, as there is insufficient evidence to make such a claim. In fact, 3c and 3d are clearly less sophisticated strategies than 3b, and at present, the relative sophistication of 5 is not known.

0.	Unable to answer question
1a.	Known basic facts
1b.	Counting techniques
1c.	*Inverse operation*
2.	Guess and check
3a.	Cover up
3b.	Working backwards
3c.	*Working backwards, then known facts*
3d.	*Working backwards, then guess and check*
4.	Formal operations/equation as object
5.	*Using a diagram*

(Based on Kieren, 1992; the strategies added from this study are in italics.)

Figure 1. Classification of strategies for solving equations

The algebra diagnostic assessment for this study is run in two parts, a knowledge section and a strategy section.

The knowledge section is administered as a written test because supplementary questions are not required. The areas investigated in this section, together with example questions, are:

- understanding of conventions and notation
  (If $n = 4$, then what is the value of $3n$?)
- understanding of the equals sign
  (Given that $7x + 4 = 15$, find the value of the □ in the equation $7x + 7 =$ □)
- understanding of arithmetic structure
  (What is the value of $18 - 12 \div 6$?)
- understanding of inverse operations
  (Replace the box with +, –, x, or ÷: If z – 27 = 25, then z = 25 □ 27)
- manipulating symbols/unknowns (acceptance of lack of closure)
  (Add 3 to $d - 1$).

The strategy section consists of a series of increasingly complex equations, which the students are asked to solve with an explanation of their thinking. There are 12 pairs of parallel questions – ones that are in context (that is, word problems), and ones that are purely symbolic. An example of a pair of parallel questions is:

- Our kapa haka group is made up of some Māori students and 11 Pākehā students. The whole group is divided into four equal-sized groups for practice sessions. Each practice group has 19 students in it. How many Māori students are there in our kapa haka group?

  $\frac{n + 12}{4} = 18.$

The increasing complexity of the equations can be illustrated by a selection of questions from the symbolic section:

- $n - 3 = 12$ (one-step equation with single digits)
- $n + 46 = 113$ (one-step equation)
- $3n - 8 = 19$ (two-step equation)
- $5n - 2 = 3n + 6$ (unknowns on both sides)
- $2n - 3 = \frac{2n + 17}{5}$ (complex structure)
- We can rearrange the equation $p = r - s$ to make r the subject, $r = p + s$.
  Similarly if $v = u + at$, then $a = ...$ (purely symbolic equation).

The questions are presented on cards so that the more difficult questions can be omitted as required without suggesting to the student that they are not coping. Each question is read to the student to minimise the impact of reading difficulties, including difficulties with reading symbolic equations. Calculators and pencil and paper are available for the students to use, but it is stressed to the students that they may use whatever method they choose.

Preliminary Results

From the interviews conducted to date, a number of points have become apparent:

- In all year levels, there are large differences between the students as to the sophistication of the strategies they are able to use. Some students are unable to use more than guess and check or counting techniques. At year 7, the majority of the students are able to use inverse operations for one-step equations, with some being able to work backwards for multi-step equations. At year 10, the majority of the students are able to work backwards for multi-step equations, with only a very small proportion being able to operate formally on equations, treating them as objects.
- The strategies the students were observed using are consistent with Kieran's (1992) classification, but they used other strategies as well. The strategy of working backwards is less homogeneous than previously reported. Many students are only just grasping this strategy and can use it only when the first step reveals a known basic fact to them for the next step. These students use the strategy of working backwards, then known facts. Other students are prevented from fully using working backwards because of a lack of knowledge of multiplication and division facts. These students use the strategy of working backwards then guess and check.
- There is a high correspondence between numeracy stage and the most sophisticated strategy a student is able to use for solving equations. Only students who are at the level of multiplicative

part–whole thinking or above are able to solve equations by working backwards or using formal operations. Numeracy stage appears to be a better predictor of the most sophisticated strategy a student is able to use than the amount of algebra teaching the student has received. There were, however, students who did not score highly on GloSS but who were able to use sophisticated strategies for solving equations. These year 9 and 10 students were invariably very efficient at using algorithms for computations and came from primary schools that did not promote numeracy.

- The most sophisticated strategy a student is able to use is extremely similar for questions that are in context or questions that are fully symbolic. In fact, many students are able to use slightly more sophisticated strategies for questions that are in context.

Discussion

Instead of looking at how hard equations are to solve and whether the students get them right, it appears to be more useful to look at what strategies the students are using to solve equations. The approach used in this study is very similar to that used in other work in the NDP, with strategy being separated out from the knowledge required for strategy use. This approach will allow the classification of the students according to their most sophisticated strategy rather than the most difficult equation they are able to solve. Within numeracy teaching, students are grouped for instruction according to their most sophisticated strategy. It is suggested that a similar approach to grouping students is likely to be beneficial for teaching students to solve equations.

As has been clearly identified (Herscovics & Linchevski, 1994), students have great difficulty with formal operations, which treat equations as objects. Few students, even from year 10, were observed using formal operations; those students who were able to use formal operations could also use the strategy of working backwards. Consistent with the perspective of Filloy and Sutherland (1996), it is suggested that these strategies are not simply alternative approaches to solving equations, but represent different stages of conceptual development. Similarly, and not surprisingly, those students who were able to work backwards could also use the strategy of inverse operations. Formal analysis of the data from this study will allow the construction of a hierarchy of sophistication of strategies.

In many current school programmes, little attention is paid to the students' numeracy stage when attempting to teach them to solve equations. The researchers found that even year 10 students, who have received at least two years' instruction in how to solve equations, are unable to use the more sophisticated strategies if they are below the numeracy stage of multiplicative part–whole thinking. Similarly, students are unable to use inverse operations for one-step equations if they are still at the numeracy stages that involve counting strategies. This strongly suggests that prerequisite numeracy should be considered when designing teaching programmes for algebra.

It would appear to be unnecessary to present students with purely symbolic equations in order to determine their most sophisticated strategy. Questions presented in context gave very similar information and were generally perceived by the students, particularly the younger ones, as being less threatening.

A number of teachers involved in the study have commented that the algebra diagnostic tool is more useful than GloSS for revealing the thinking of students at the upper end of the Number Framework. This is because GloSS focuses on mental strategies (and does not value the use of algorithms), while the algebra tool has a focus on students' understanding of arithmetic structure and the nature of equations.

The preliminary data presented in this paper suggest that the approach adopted is revealing useful information about students' understanding of aspects of algebra. It is anticipated that formal data analysis will establish a hierarchy of strategies, make explicit the connections between numeracy and algebraic strategies, and clarify the role of contexts.

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