Mathematics in the New Zealand Curriculum
Second Tier

Strand: Number and Algebra Thread: Patterns and Relationships Level: Four

Achievement Objectives:
- Generalise properties of multiplication and division with whole numbers.
- Use graphs, tables, and rules to describe linear relationships found in number and spatial patterns.

Exemplars of student performance:

Exemplar One:

Isabel (student) solves equations involving multiplication and division with unknowns on either side of the equals sign. She is able to solve the equations relationally. This means that she applies invariant number properties rather than calculates the total value for each side of the equation. She understands the equals sign as a statement of balance that preserves under any transformation of the terms in the equation.

To solve:

a. $(67 \times 8) + (67 \times 40) = \square \times 67$: Isabel notices that the commutative property shows $\square \times 67 = 67 \times \square$ and the distributive property shows $(67 \times 8) + (67 \times 40) = 67 \times 48$. Therefore $\square = 48$.

b. $2 \times 2 \times \square \times 67 = 12 \times 67$: Isabel realizes that $2 \times 2 \times \square$ must have a product of 12 so $\square = 3$.

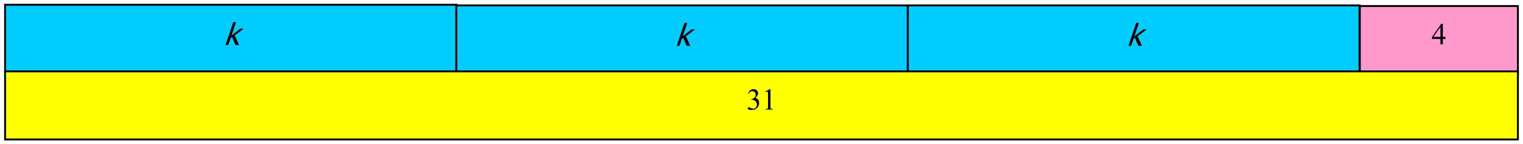
c. $462 \div 6 = \square \div 3$: Isabel recognizes that to preserve the equality both the dividend (number divided) and the divisor must be halved. Therefore $\square = 462 \div 2 = 231$.

d. $399 \div 7 = (\square \div 7) + (350 \div 7)$: Isabel sees by the distributive property that $350 + \square = 399$ so $\square = 49$.

Isabel's responses show achievement at level four because she thinks relationally about equations involving multiplication and division. This shows that she understands the key properties of multiplication and division, the commutative property (e.g. $13 \times 8 = 8 \times 13$ but $48 \div 6 \neq 6 \div 48$), the distributive property (e.g. $24 \times 36 = (20 \times 36) + (4 \times 36)$ and $201 \div 3 = (180 \div 3) + (21 \div 3)$), and the associative property (e.g. $2 \times (3 \times 5) = (2 \times 3) \times 5$).

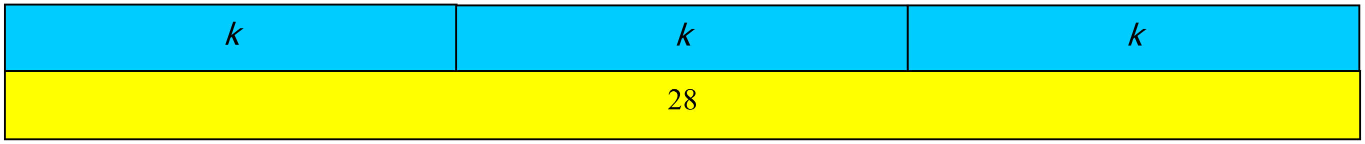
Exemplar Two:

Henare has this diagram, and solves for k.

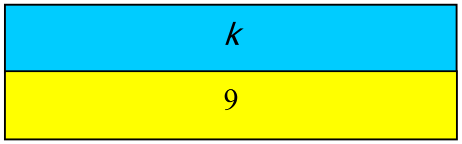


He writes the linear equation, $3k + 4 = 31$, and operates on both sides of the equals sign using inverse operations. He is able to represent each step using equations and the strip diagram.

$3k = 27$ (subtracting 4)

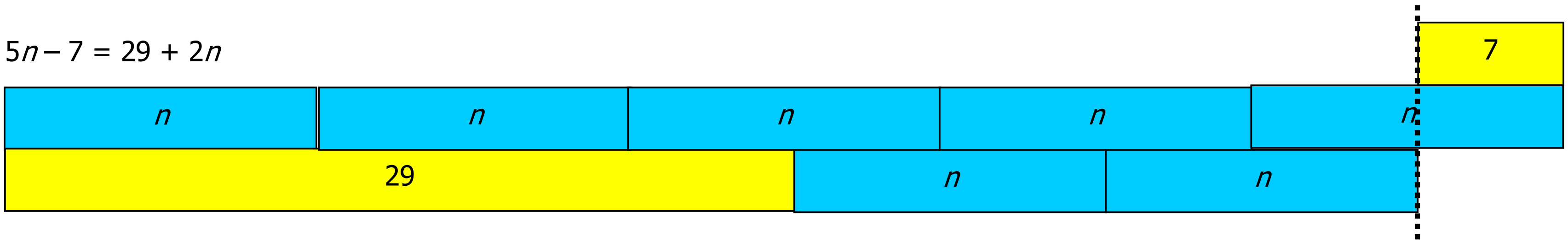


$k = 9$ (dividing by 3)

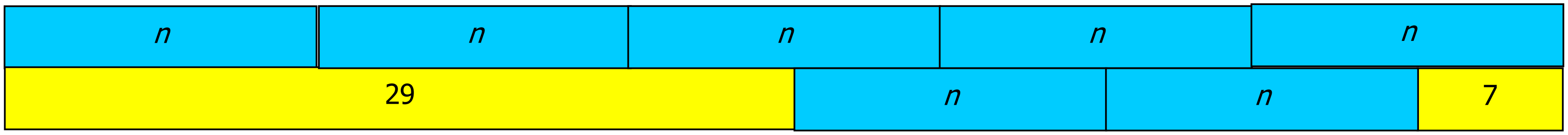


Henare solves complex linear equations with the support of strip diagrams.

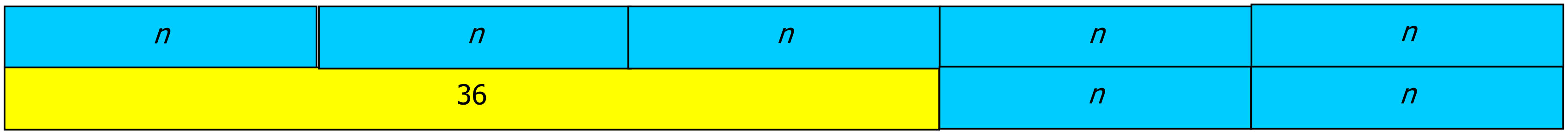
$5n - 7 = 29 + 2n$



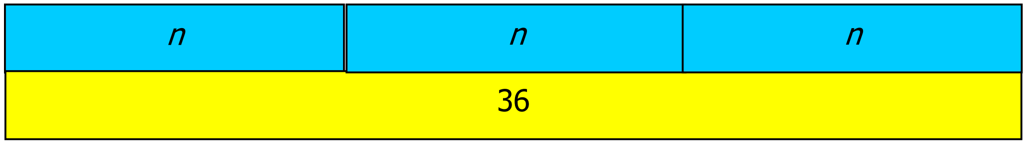
$5n = 29 + 2n + 7$ (adding 7)



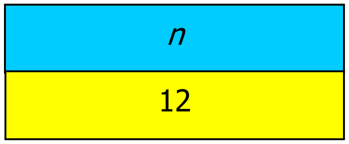
$5n = 36 + 2n$ (combining terms)



$3n = 36$ (subtracting $2n$)

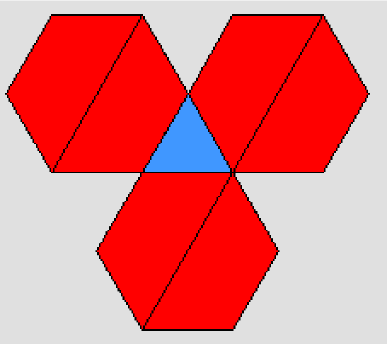


$n = 12$ (dividing by 3)



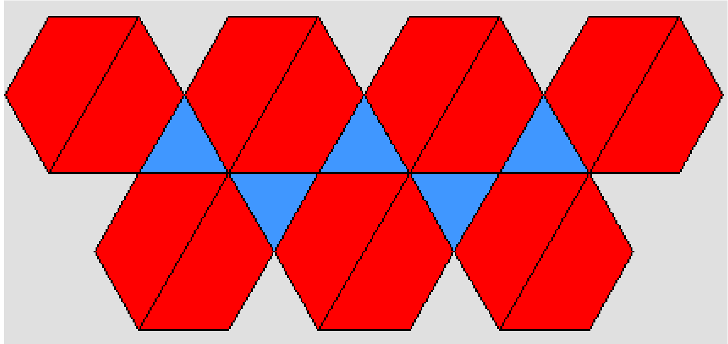
Henare's achievement exemplifies level four because he is able to solve linear equations by applying inverse operations. He understands that equality of an equation preserves if the same operation is performed on both sides. Henare recognizes that subtraction undoes addition and vice versa, and division undoes multiplication and vice versa. These are inverse operations.

Exemplar Three:
Mei-Ling, Ofisa and Whatu work out the number of triangles required to match with 42 trapeziums in this paving pattern.



1 triangle…

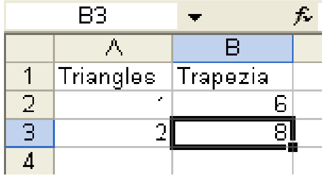
built on to give…



5 triangles

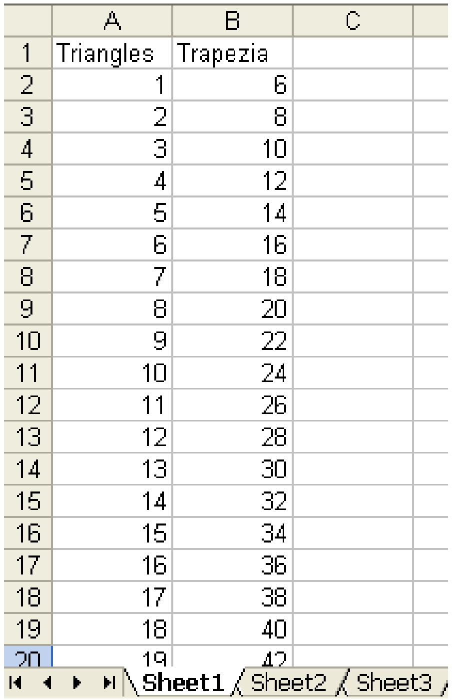
a. Mei-Ling notices that six trapeziums surround the first triangle and each triangle after that needs two trapeziums. She calculates 42 – 6 = 36 to find the number of trapeziums available after surrounding the first triangle. She calculates 36 ÷ 2 = 18. This gives the number of additional triangles surrounded. 18 + 1 = 19 is the total number of triangles required.

b. Ofisa uses a computer spreadsheet to find the required ordered pair. He also sees that the number of trapeziums increase by two for every extra triangle. He uses this additive relationship to enter an "add two to the number above" rule into B3 and uses the fill down function to extend the pattern.

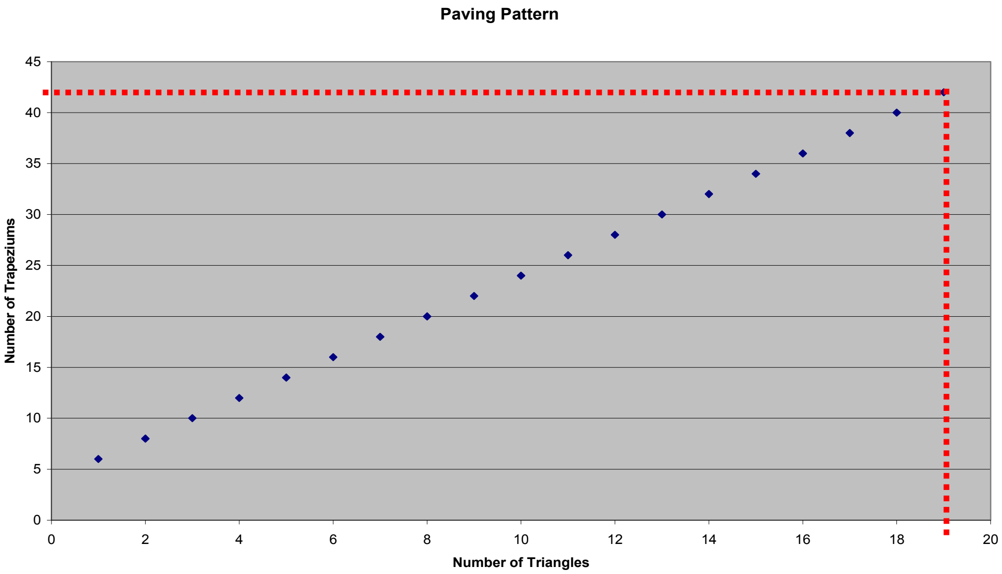


	A	B
1	Triangles	Trapezia
2	[illegible]	6
3	2	8
4		

Ofisa uses the spreadsheet graphing function to create a scatter plot. He knows that the relation is linear because the number of hexagons increases by a regular number, two, for every extra triangle. Ofisa reads the ordered pair (19, 42) from the graph. This shows that 42 hexagons (y-axis) match 19 triangles (x-axis).



	A	B	C
1	Triangles	Trapezia	
2	1	6	
3	2	8	
4	3	10	
5	4	12	
6	5	14	
7	6	16	
8	7	18	
9	8	20	
10	9	22	
11	10	24	
12	11	26	
13	12	28	
14	13	30	
15	14	32	
16	15	34	
17	16	36	
18	17	38	
19	18	40	
20	19	42	



c. Whatu creates a linear equation for the relation between triangles and hexagons. She records $h = 2t + 4$ from her knowledge that each extra triangle matches two extra hexagons and that this pattern requires an additional four hexagons to start. She writes:

$$42 = 2t + 4$$
$$38 = 2t$$
$$19 = t$$

Mei-Ling, Ofisa and Whatu all show level four achievement as they use strategies to find missing values in a linear relation. Mei-ling and Whatu apply multiplicative thinking to create rules for predicting further pairs in the relation. Mei-Ling recognizes that the number of hexagons is six for the first triangle and two for every extra triangle (This could be written as $h = 6 + 2\ (t - 1)$. Whatu uses a linear equation to solve the missing value problem showing that she understands inverse operations. Ofisa recognizes the linearity in the relation and uses technology to find the set of ordered pairs.

Exemplar Four:

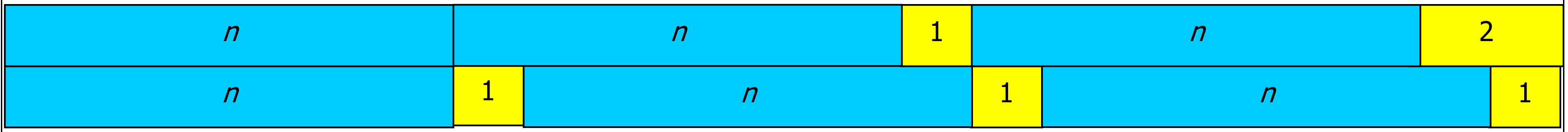
Zac investigates this pattern of equations:

$$7 + 8 + 9 = 3 \times 8$$
$$3 + 4 + 5 = 3 \times 4$$
$$9 + 10 + 11 = 3 \times 10$$

He records other equations that he believes fit the same pattern, e.g. $99 + 100 + 101 = 3 \times 100$. Zac uses the letter n to represent the first number in each equation and writes a general rule for the equation set;

$$n + (n + 1) + (n + 2) = 3 \times (n + 1)$$

He sets up a strip model to explain why the equation set works:



Zac notices that, by grouping the n's and the ones, each side of the equation could be rewritten as $3n + 3$.

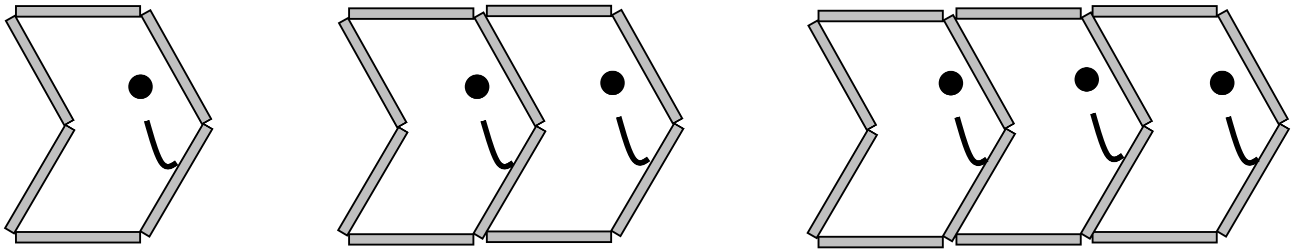
Zac's achievement exemplifies level four because he recognizes that variables (n's) can be combined in a multiplicative way. He generalizes the relationship between repeated addition and multiplication and applies the distributive property in seeing that $3(n + 1) = 3n + 3$. In doing so he accepts lack of closure, in that the relationships hold no mater what value is chosen for n.

Important teaching ideas (working at):

The focus of algebra at level 4 is generalization of the multiplicative number properties, application of these properties to relations, and emerging competence in expressing these properties using the language of variables. Generalization involves these progressions (Mason, 2003):

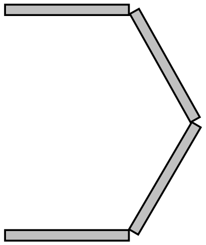
1. **Gazing at the whole**

   For example, a student is shown this growing pattern of faces made from matchsticks. The pattern captures their attention.



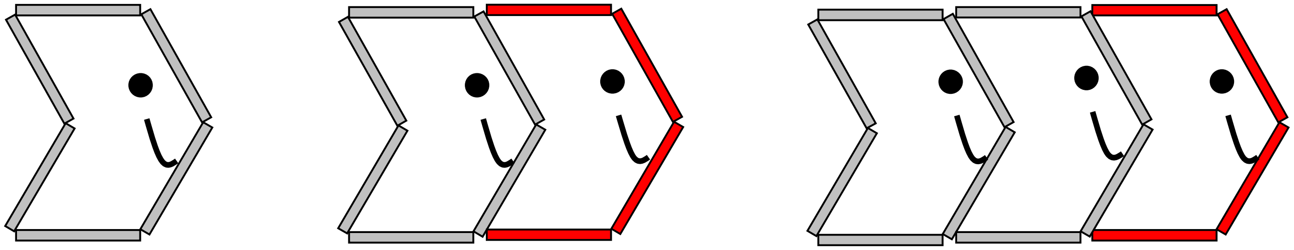
2. **Discerning details**

   The student recognizes that the pattern is composed of a repeating element.
   He or she notices that the first face has two extra matchsticks.



3. **Recognizing relationships**

   The student notices that the next pattern in the sequence has four more matchsticks than the previous pattern:



4. Perceiving properties

The student sees that a multiplicative relationship exists between the number of faces and the number of matchsticks needed. Since four matchsticks are added each time the relationship is linear and involves a "times four" characteristic. Since the first face took six matchsticks, two extras must be added to compensate

Number of faces	1	2	3	4		10		100	$\times 4 + 2$
Number of matchsticks	6	10	14	18		42		402	

$+4$ $+4$ $+4$

He or she expresses the relation using a linear equation, $m = 4f + 2$, where m represents the number of matchsticks and f the number of faces.

5. Reasoning on the basis of the properties alone

This reasoning is expected at later levels of the curriculum as it involves using generalizations to derive other generalizations without the need to fall back to the original situations. (See levels 5 and 6 for examples)

From their work on number strategies students will have access to multiplicative properties without consciously thinking about these properties as generalizations. Require students to interpret the number strategies of others, leading to finding unknowns in equations without calculating, then expressing the strategies as generalizations using letter symbols as appropriate. Note that student need to interpret the "equals sign", =, as a statement of balance rather than as an indicator that the answer follows, as in 4 + 5 = ☐. Examples illustrating the key properties of the multiplicative field follow:

Multiplicative Property	Given strategy	Missing unknown	Algebraic substitution (Extension Only)	Algebraic Generalization (Extension Only)
Commutative Multiplication	R solves 999 × 3 = ☐ as 3 × 999 = ☐. How would R solve 244 × 4 = ☐?	What number does Δ represent in each equation? 56 × 43 = 43 × Δ Δ × 52 = 52 × 87	What would go in each ☐ to make each equation balance? 45 × k = ☐ × 45 ☐ × 63 = 63 × d	$a \times b = b \times a$ or $ab = ba$

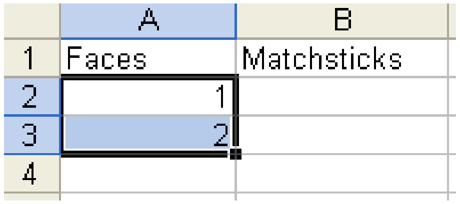
Multiplicative Property	Given strategy	Missing unknown	Algebraic substitution (Extension Only)	Algebraic Generalization (Extension Only)
Distributive Multiplication	T solves 47 × 6 = ☐ as (40 × 6) + (7 × 6) = ☐. How would T solve 52 × 8 = ☐?	What number does Δ represent in each equation? 9 × 58 = (9 × Δ) + (9 × 8) Δ × 4 = (30 × 4) + (7 × 4)	What would go in each ☐ to make each equation balance? 75 × ☐ = (70 × y) + (5 × *y*) *q* × 28 = (☐ × 28) + (6 × 28)	$a \times b = (a \times c) + (a \times d)$. or $ab = ac + ad$, where $c + d = b$
Distributive Division	M solves 96 ÷ 4 = ☐ as (80 ÷ 4) + (16 ÷ 4) = ☐. How would M solve 96 ÷ 3 = ☐?	What number does Δ represent in each equation? 441÷ 7 = (Δ ÷ 7) + (21÷ 7) Δ ÷ 9 = (1800 ÷ 9) + (81 ÷ 9)	What would go in each ☐ to make each equation balance? 385÷ 5 = (350÷ 5) + (☐ ÷ 5) ☐ ÷ 6 = (540 ÷ 6) + (36 ÷ 6)	$a \div b = (c \div b) + (d \div b)$, or $\frac{a}{b} = \frac{c}{b} + \frac{d}{b}$, where $c + d = a$
Associative Multiplication	P solves 7 × 8 × 5 = ☐ as 7 × 40 = ☐. How would T solve 5 × 9 × 6 = ☐?	What number does Δ represent in each equation? 9 × 7 × 4 = 4 × Δ × 7 3 × Δ × 8 = 8 × 3 × 5	What would go in each ☐ to make each equation balance? 3 × *t* × 33 = 99 × ☐ 25 × 6 × ☐ = *z* × 150	$(a \times b) \times c = a \times (b \times c)$ or $abc = bac = cba$, etc.
Inverse Operations	W checks 75 × 4 = 300 by 300÷ 4 = 75. How would T check 392÷ 7 = 56?	What number does Δ represent in each equation? 34 × 9 = 306 so 306÷ Δ = 9 402÷ 6 = 67 so 67 × Δ = 402	What would go in each ☐ to make each equation balance? 8 × *p* = 296 so 296 ÷ 8 = ☐ 731 ÷ 43 = *h* so 43 × ☐ = 731	$ab = c$ so $\frac{c}{a} = b$ or $\frac{c}{b} = a$

Relations are sets of ordered pairs, e.g. (2,7). The first value in an ordered pair represents a possible value of the independent variable, i.e. the variable you are free to manipulate. The second value represents the corresponding value of the dependent variable, i.e. the variable that's value is determined by the value of the independent variable. In the matchstick pattern above, the number of faces was the independent variable and the number of matchsticks was the dependent variable. Relations are graphed on a number plane with the values of the independent variable on the x-axis (across) and the values of the dependent variable on the y-axis (vertical). In mathematics, we tend to be more interested in relations that show a graphical pattern than those that don't.

At level four students need to look for both recursive (repeating) and direct relationships in sets of ordered pairs. Recursive relationships involve looking at what happens to a term in a sequence to get the next term. A direct relationship links the values of the independent variable to those of the dependent variable. Tables and graphs are important representations for looking at these relationships. Use computer technology whenever possible to speed up the creation of these representations and allow students to see how changes in table values affect the corresponding graph.

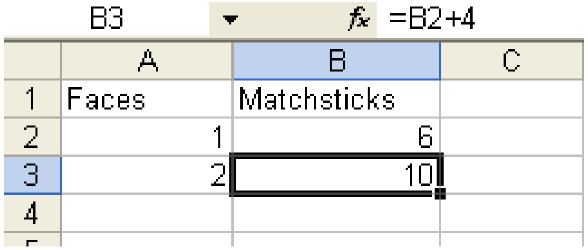
The examples below use the faces matchstick pattern as a context.

Set up the table columns as below. Use the fill down function to generate the x-values. Click on the bottom right corner of the blocked area and scroll down the column with the left button down. This fill the sequence of counting numbers into column A.



	A	B
1	Faces	Matchsticks
2	1	
3	2	
4		

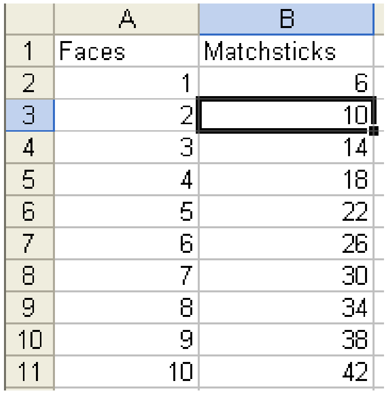
Type in the first y-value, 6, in cell B2. In cell B3 type the function =B2+4 then hit return. This will take the value in B2 add two to it, and put the answer in B3.



B3 fx =B2+4

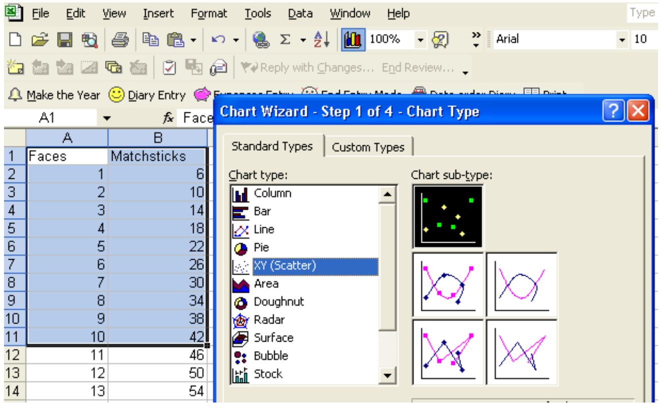
	A	B	C
1	Faces	Matchsticks	
2	1	6	
3	2	10	
4			

By clicking on the bottom right corner of cell B3 and scrolling down with the left mouse key on will fill the "add four to the cell value above" pattern, as shown.



	A	B
1	Faces	Matchsticks
2	1	6
3	2	10
4	3	14
5	4	18
6	5	22
7	6	26
8	7	30
9	8	34
10	9	38
11	10	42

Block the whole table using your mouse, holding down the left button. Click the graphing icon (bar graph) in the toolbar. Choose "Scatter plot" to graph the relation.



A1 fx Face

Chart Wizard - Step 1 of 4 - Chart Type

Chart type: XY (Scatter)

	A	B
1	Faces	Matchsticks
2	1	6
3	2	10
4	3	14
5	4	18
6	5	22
7	6	26
8	7	30
9	8	34
10	9	38
11	10	42
12	11	46
13	12	50
14	13	54

Click next on the Chart Wizard until this screen appears. Type in the title, and x and y values. Click off the legend.

Continue clicking next in the Wizard. You will get a choice of placing The graph into the existing sheet (as shown) or creating it as a separate sheet. For presentation a separate sheet is best. For continued interaction with the table choose the default

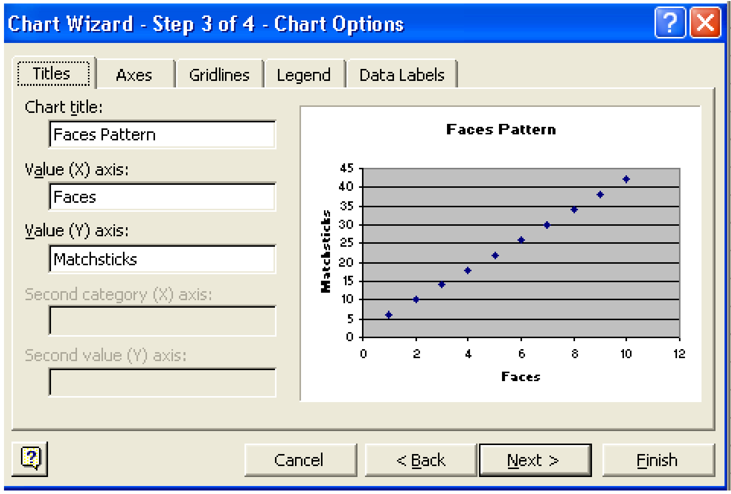


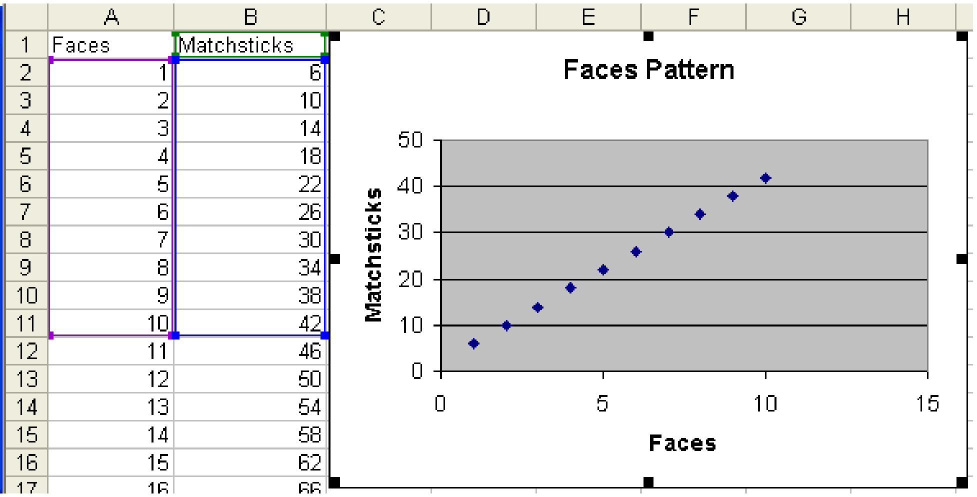
Chart title: Faces Pattern

Value (X) axis: Faces

Value (Y) axis: Matchsticks

Second category (X) axis:

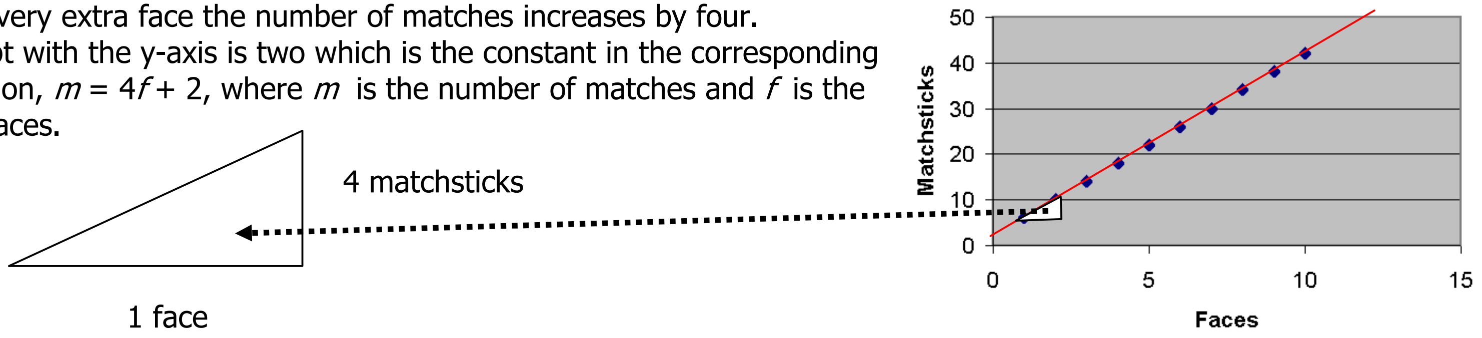
Second value (Y) axis:



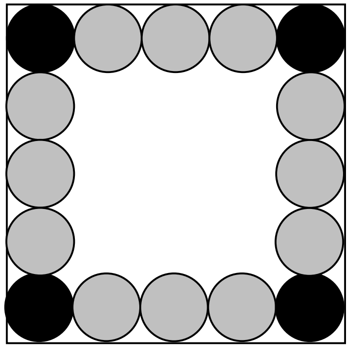
	A	B
1	Faces	Matchsticks
2	1	6
3	2	10
4	3	14
5	4	18
6	5	22
7	6	26
8	7	30
9	8	34
10	9	38
11	10	42
12	11	46
13	12	50
14	13	54
15	14	58
16	15	62
17	16	66

If the graph is embedded within the table sheet the values in the table can be changed and the graph will alter accordingly. It is important for students to realize that this pattern is linear. The ordered pairs lie on a line. The slope of the line is four. This means for every extra face the number of matches increases by four.

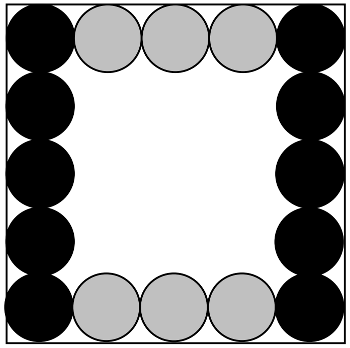
The intercept with the y-axis is two which is the constant in the corresponding linear equation, $m = 4f + 2$, where m is the number of matches and f is the number of faces.



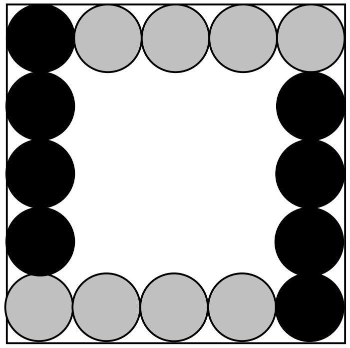
The three main contexts to encounter relations in are geometric, numeric and statistical (measurement). Contexts can be a combination of these. Geometric pattern has the advantage of allowing students to take advantage of both spatial and numeric reasoning. For example, consider this border pattern that may be found on table mats. Students attendance to spatial grouping of the circles affects how they answer the question, "How many circles would be in a border pattern 20 circles across?"



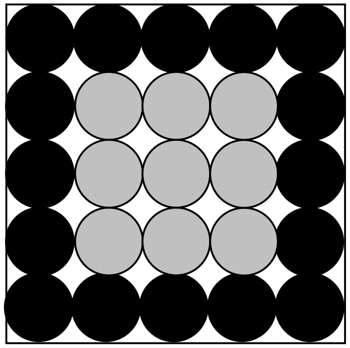
Wiremu see this…



Vey-Un sees this…



Tabitha sees this…



Maria sees this.

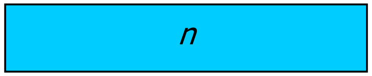
Their calculations for the pattern 20 circles across might be:

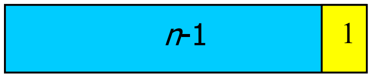
(4 × 18) + 4 = 76	(2 × 20) + (2 × 18) = 76	4 × 19 = 76	(20 × 20) – (18 × 18) = 76

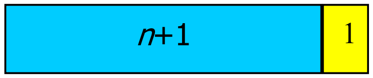
The algebraic expression for each view, with *n* representing the number of circles on the bottom row is:

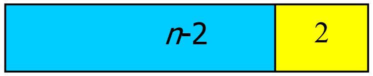
$4(n-2)+4$	$2n+2(n-2)$	$4(n-1)$	$n^2-(n-2)^2$

Compare equivalent expressions for the same relation structurally. Firstly, if they produce the same values of the dependent variable then they are usually equivalent, but not always. Secondly, using strip or cups and counters models show the structural equivalence:



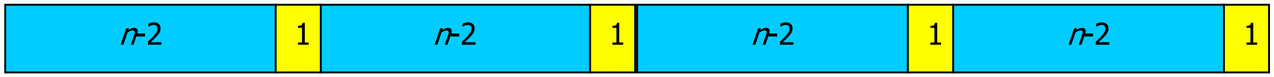


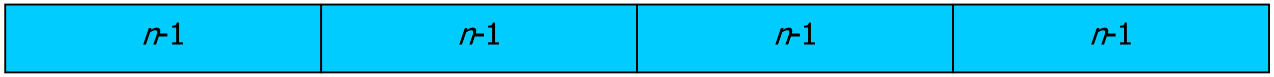




$4(n-2) + 4$ can be shown as:

Re-arrange expression by splitting the four into ones and distributing them to change the n-2 terms into n-1 terms.

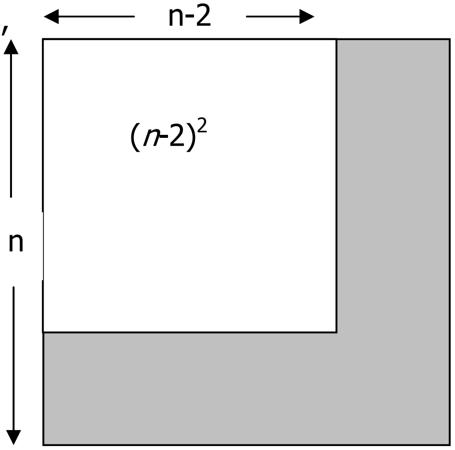


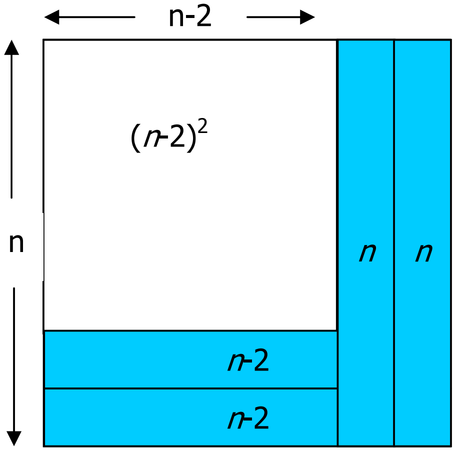


So $4(n-2) + 4 = 4(n-1)$
Thirdly, students operate on the symbols themselves,
e.g. $2n + 2(n - 2) = 2n + 2n - 4$
$= 4n - 4$
$= 4(n-1)$

Note that $n^2 - (n-2)^2$ can be modeled as:

The grey shaded area is equal to 2n + 2(n-2) or 4n – 4 = 4(n-1).





Numeric patterns have the advantage of minimizing noise from the problem contexts. Equation patterns and function machines provide useful vehicles for relational thinking.
Below are some examples of equation patterns:

1 + 3 = 4	2 × 9 = (1 × 9) + (1 × 9)	4 ÷ 2 = 2
2 + 6 = 8	4 × 9 = (2 × 9) + (2 × 9)	8 ÷ 4 = 2
3 + 9 = 12	6 × 9 = (3 × 9) + (3 × 9)	12 ÷ 6 = 2

Ask the students to:

1. Write the next two equations in the pattern.
2. Write an equation that is "a long way down" in the pattern.
3. Write a rule for all equations in the pattern.

For example, in the pattern beginning as 1 + 3 = 4, students might write 4 + 12 = 16 and 5 + 15 = 20 as the next equations. In doing so they notice that the left-hand addends increase by one and three respectively, and that the second addend is three times the first addend. However, the right-hand sum is usually calculated. To write an equation further down they may choose any whole number to start the equation, e.g. 16 + 48. They then need to find the relationships across the equals sign to complete their equation:

$$\underbrace{\overbrace{16 + 48}^{\times 3} = 64}_{\times 4}$$

Expressing a general rule for the equation set requires the use of some symbol to represent the first addend in the equation. Suppose *a* represents the first addend then the general rule can be written as:

$$a + 3a = 4a$$

Function machine contexts involve the creation of ordered pairs. Numbers are entered into a hypothetical machine that performs a consistent transformation on them to produce output numbers. Students are asked to find the algorithm (rule) that the machine is using. It is important that the numbers entered are not in sequence if direct, multiplicative rules are to be encouraged. Below are some examples:

In	Out
7	21
3	12
11	33
5	15
n	$3n$

In	Out
8	4
11	5.5
24	12
3	1.5
n	$n \div 2$

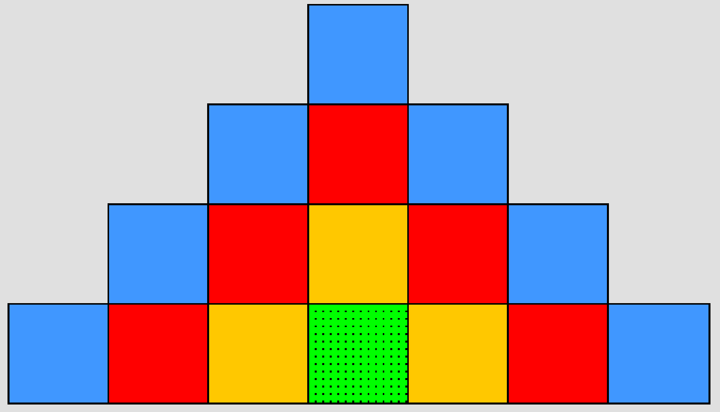
In	Out
6	21
3	9
10	37
8	29
n	$4n-3$

In	Out
5	5.5
10	8
7	6.5
2	4
n	$\frac{1}{2}n+3$

In	Out
5	25
9	81
1	1
11	121
n	n^2

In	Out
3	8
2	4
0	1
6	64
n	2^n

As the students become proficient it is important that they are exposed to a variety of rules including linear relations with fractional factors, e.g. $\frac{3}{4}n + 2$, exponents, e.g. 2^n, and simple quadratics, e.g. $n^2 + 3$. Exposure to non-linear relations is important in helping students understand that a broader range of relations exists. It also requires them to make critical choices about whether they can find direct rules or they need to use recursive rules. For example, students have to find out how many square tiles make up a staircase ten squares high in this pattern:

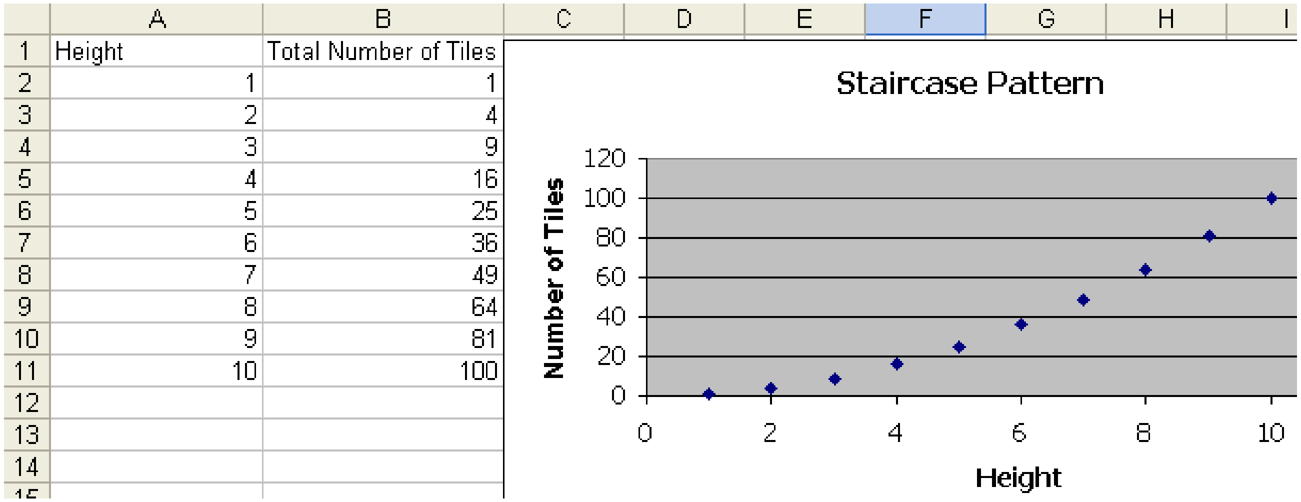


The block shading may help them to realize how many tiles are added to create each additional layer of the staircase and that nine extra tiles make the staircase five high. Organizing the results into a table in sequence is likely to lead to use of a recursive rule:

Layers high	1	2	3	4	5					
Total number of tiles	1	4	9	16	25					

+3 +5 +7 +9

The recursive rule that the differences are increasing by two tiles each times is a powerful observation that, with the use of a spreadsheet, gives the number of tiles needed to build pyramids of any reasonable height. Graphing this relation will show that it is not linear since the differences, and therefore the slope of the graph, are not constant.



	A	B
1	Height	Total Number of Tiles
2	1	1
3	2	4
4	3	9
5	4	16
6	5	25
7	6	36
8	7	49
9	8	64
10	9	81
11	10	100
12		
13		
14		

In the absence of technology the students may look for a direct rule, in this case $t = h^2$, where t is the number of tiles and h is the layers high. In the case of more difficult quadratic and exponential relations recursive rules are all that is expected for achievement at level four.

Useful resources

Figure It Out (Learning Media)
Algebra Level 3-4, pages 1-24
Algebra year 7/8 Books 2 and 3
Number Sense and Algebraic Thinking, Levels 3-4
For full references and learning outcomes for the Figure It Out books go to nzmaths:
http://www.nzmaths.co.nz/numeracy/PlanLinks/AlgPlannerAA-AM.pdf
[Comprehensive Teacher notes are provided for each student book. These notes have been distributed to schools and can also be accessed through http://www.tki.org.nz/r/maths/curriculum/figure/index_e.php

Numeracy Project Book 9: Teaching Number through Measurement, Geometry, Algebra, and Statistics, pages 30-40.
nzmaths.co.nz units (This website is sponsored by the Ministry of Education)
http://www.nzmaths.co.nz/algebra/Units/fourfourschallenge.aspx
http://www.nzmaths.co.nz/algebra/Units/youcancountonsquares.aspx
http://www.nzmaths.co.nz/algebra/Units/balancingacts.aspx
http://www.nzmaths.co.nz/algebra/Units/matidaswaltz.aspx
http://www.nzmaths.co.nz/algebra/Units/twocompany.aspx
http://www.nzmaths.co.nz/algebra/Units/GeneralRule.aspx
http://www.nzmaths.co.nz/algebra/Units/truthtrianglessquares.aspx
http://www.nzmaths.co.nz/algebra/Units/tukutukupanels.aspx

Digital Learning Objects (These are accessed through the Ministry of Education Digi-Store and are the result of a collaborative project run by The Learning Federation, Australia)
http://www.nzmaths.co.nz/LearningObjects/A4.aspx

Other Website links:
http://illuminations.nctm.org/ActivityDetail.aspx?ID=33
http://nlvm.usu.edu/en/nav/category_g_3_t_2.html
http://www.bbc.co.uk/education/mathsfile/gameswheel.html